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The Constant Error Curve Problem for Varisolvent Families

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Our main theorem states that under a certain existence hypothesis a varisolvent family does not permit a best approximation with a constant error. We deal with real valued continuous functions on a compact real interval using the Chebyshev (uniform) norm.

Our result is applied to simultaneous approximation to show that a constant error cannot arise there. Further topics such as restricted range approximation, a betweenness property, and approximation on a proper compact subset of an interval are also studied.

The existence hypothesis of the main theorem appears to be satisfied by most known varisolvent families. Examples are given.

1. INTRODUCTION

Let $[a, b]$ be a nondegenerate compact real interval, and let $C[a, b]$ be the set of all real valued continuous functions on $[a, b]$. By R^s we mean Euclidean s-space. Let P be a given nonempty subset of R^s. Our approximating family will be $\mathscr{V} = \{F(A, x) \mid A = (a_1, \ldots, a_s) \in P, x \in [a, b]\}$. For each fixed $A \in P$, $F(A, x) \in C[a, b]$ and $\| A \| = \max_{1 \leqslant i \leqslant s} | a_i |$.

We now define Property Z and local solvency. Let n be a positive integer.

DEFINITION 1. $F(A, x) \in \mathscr{V}$ has Property Z of degree n on $[a, b]$ if for any $F(A_1, x) \in \mathscr{V}$ with $F(A_1, x) \not\equiv F(A, x)$, $F(A_1, x) - F(A, x)$ has at most $n - 1$ zeros on $[a, b]$.

DEFINITION 2. $F(A, x) \in \mathscr{V}$ is locally solvent of degree n on $[a, b]$ if:

(1) given n distinct points in $[a, b]$, $\{x_i\}_{i=1}^n$, $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$ and

(2) given $\epsilon > 0$, there exists a $\delta = \delta(A, \epsilon, x_1, \ldots, x_n) > 0$ such that for any set of real numbers $\{z_i\}_{i=1}^n$ with $| z_i - F(A, x_i)| < \delta$ for $1 \leqslant i \leqslant n$, there exists $F(A_1, x)$ in $\mathscr{V}$ satisfying

(3) $F(A_1, x_i) = z_i$, $1 \leqslant i \leqslant n$, and

(4) $\max_{x \in [a,b]} |F(A_1, x) - F(A, x)| < \epsilon$.

DEFINITION 3. $F(A, x) \in \mathscr{V}$ is a varisolvent function of degree n at A if (1) $F(A, x)$ has Property Z of degree n, (2) $F(A, x)$ is locally solvent of degree n, and (3) $F(A, x)$ is not locally solvent of degree $n + 1$.

We denote the degree of $F(A, x)$ by $m(A)$. $\mathscr{V}$ is said to be a varisolvent family if each element of $\mathscr{V}$ is a varisolvent function.

An important feature of varisolvent families is that the degree, $m(A)$, may vary with A. Although not required in the definition of varisolvency, the degrees of the functions in $\mathscr{V}$ are stipulated to be uniformly bounded (see [15, p. 4]). We shall assume that $m(A) \leqslant M$ for all $A \in P$ where M is some positive real number. We choose M such that there exists an $A \in P$ with $m(A) = M$.

For unisolvent families the degree does not vary.

DEFINITION 4. $\mathscr{U}$ is said to be a unisolvent family of degree n on $[a, b]$ if:

(1) (Zero Property) each $F(A, x) \in \mathscr{U}$ has Property Z of degree n and

(2) (Solvency) given $\{x_i\}_{i=1}^n$ distinct points, $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$, and any set of real numbers $\{z_i\}_{i=1}^n$, there exists $F(A, x) \in \mathscr{U}$ with $F(A, x_i) = z_i$ for $1 \leqslant i \leqslant n$.

The degree of each $F(A, x) \in \mathscr{U}$ is $m(A) = n$. A unisolvent family is in fact varisolvent.

For later comment, we define the concept of a Haar space.

DEFINITION 5. Let H be an n-dimensional subspace of $C[a, b]$. H will be called an n-dimensional Haar space if any nontrivial element of H has at most $n - 1$ distinct zeros in $[a, b]$.

Definition of the Approximating Problem

Let X be any nonempty compact subset of $[a, b]$ and let $C(X)$ be the set of real valued continuous functions on X. By the Chebyshev norm of $g \in C(X)$ we mean $\max_{x \in X} |g(x)|$, denoted by $\|g(x)\|_X$. By $\|g(x)\|$ we mean $\max_{x \in [a,b]} |g(x)|$ where $g \in C[a, b]$.

The approximating problem is to approximate any given $f \in C(X)$ by elements of a varisolvent family $\mathscr{V}$ in the Chebyshev norm. We seek a best approximation in the following sense. $F(A^*, x)$ is said to be a best approximation on X to $f \in C(X)$ from $\mathscr{V}$ if $\|f(x) - F(A, x)\|_X = \inf_{A \in P} \|f(x) - F(A, x)\|_X$. We will assume that the function to be approximated, f, is such that $f(x) \neq$

$F(A, x)$ on X for any $A \in P$. For each $A \in P$ define $e(A) = \|f(x) - F(A, x)\|_X$. Note that our assumption implies $e(A) > 0$ for all $A \in P$.

For $F(A, x)$ in $\mathscr{V}$, $E(A, x) = f(x) - F(A, x)$ is called the error function (curve) with respect to f, associated with $F(A, x)$. Frequently best approximations are characterized by alternation of their error curves.

DEFINITION 6. $E(A, x)$ is said to alternate n times on X if there exist $n + 1$ distinct points in X, $\{x_i\}_{i=1}^{n+1}$, $a \leqslant x_1 < x_2 < \cdots < x_{n+1} \leqslant b$, such that:

(1) $|f(x_i) - F(A, x_i)| = \|f(x) - F(A, x)\|_X$ for $1 \leqslant i \leqslant n + 1$ and

(2) $[f(x_i) - F(A, x_i)] = -[f(x_{i+1}) - F(A, x_{i+1})]$ for $1 \leqslant i \leqslant n$.

Those x_i for which $E(A, x_i) = \|f(x) - F(A, x)\|_X$ are called lower alternation points and those for which $E(A, x_i) = -\|f(x) - F(A, x)\|_X$ are called upper alternation points.

In 1961, Rice [13] presented the following theorem (the version below appears in [15, p. 10]).

THEOREM 1 (Alternation Theorem). *Let $\mathscr{V}$ be a varisolvent family on X and $f \in C(X)$. Then $F(A^*, x)$ is a best approximation to f from $\mathscr{V}$ on X iff $f(x) - F(A^*, x)$ alternates at least $m(A^*)$ times on X.*

A similar result for unisolvent families was given in 1950 by Tornheim [17].

Definition and History of the Constant Error Curve Problem

DEFINITION 7. A varisolvent family $\mathscr{V}$ on $[a, b]$ is said to permit a constant error on X if there exists $f \in C(X)$ and a best approximation $F(A^*, x)$ to f from $\mathscr{V}$ on X such that $f(x) - F(A^*, x) \equiv C$ on X; C a nonzero constant. Further, we say that $F(A^*, x)$ gives rise to a constant error curve with respect to f.

In 1968, Dunham pointed out that the proof of Theorem 1 and the proof of Tornheim's result omitted the possibility of a constant error curve (see [6]). Since that time, the gap in the proof of Tornheim's result has been filled by Barrar and Loeb (see [1]). In the same paper Barrar and Loeb show, for varisolvent families, that if the degree of the best approximation is 1, 2, or 3, then it may not give rise to a constant error curve with respect to any $f \in C[a, b]$. Various approaches have been tried to complete the argument for the varisolvent case.

The approach taken by Meinardus and Schwedt, and by Barrar and Loeb, is the following. Let P be an open nonempty subset of real Euclidean s space. Let $\mathscr{F} = \{F(A, x) \mid A = (a_1, \ldots, a_s) \in P,\ x \in [a, b],\ F(A, x) \text{ real valued}\}$ be a given approximating family. The following assumptions are made.

(1) Each $F(A, x)$ in $\mathscr{F}$ is continuous in A and x.

(2) For each i, $1 \leqslant i \leqslant s$, $\partial F(A, x)/\partial a_i$ exists and is continuous in A and x.

(3) For each $F(A, x)$ in $\mathscr{F}$, the span of the functions $H(A) \doteq \{\partial F(A, x)/\partial a_i \mid 1 \leqslant i \leqslant s\}$ is a Haar space of dimension $d(A) \geqslant 1$.

(4) Each $F(A, x)$ in $\mathscr{F}$ has Property Z of degree $d(A)$. Using these assumptions, Barrar and Loeb [2, p. 595] prove that a constant error may not occur for such a family. Some varisolvent families satisfy these requirements where for $F(A, x)$ in $\mathscr{F}$, $d(A) = m(A)$. Thus, they do not permit a constant error.

Meinardus, Taylor and Braess have made other contributions. Meinardus and Taylor have shown the following theorem which appears in [16].

THEOREM 2. *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$. Assume there exists an extension $[a_1, b_1]$ of $[a, b]$, (either $-\infty < a_1 < a$ or $b < b_1 < \infty$ or both), and a varisolvent family $\mathscr{V}_1$ on $[a_1, b_1]$ such that $\mathscr{V}_1$ restricted to $[a, b]$ is $\mathscr{V}$. Then $\mathscr{V}$ does not permit a constant error on $[a, b]$.*

Braess has eliminated the possibility of a constant error curve for varisolvent families where the degree of the best approximation is maximal (see [3]). Maximal of course means, if $F(A^*, x)$ is the best approximation, then $m(A^*) = M$ where M is the uniform bound for the degrees of the functions in $\mathscr{V}$.

2. PRELIMINARIES

We present a series of definitions and lemmas used in the proof of the main theorem.

DEFINITION 8. Let $g \in C[a, b]$. Then x_0 is said to be a simple zero of g (a zero of multiplicity 1) on $[a, b]$ if $g(x_0) = 0$ with either (1) $x_0 = a$ or $x_0 = b$ or (2) $x_0 \in (a, b)$ such that there exists a neighborhood of x_0, $N(x_0)$, with the property that for all x_1, x_2 in $N(x_0)$ with $x_1 < x_0 < x_2$, $g(x_1)\, g(x_2) < 0$.

DEFINITION 9. Let $g \in C[a, b]$. Then x_0 is said to be a double zero of g (a zero of multiplicity 2) on $[a, b]$ if $g(x_0) = 0$, $x_0 \in (a, b)$ and if there exists a neighborhood of x_0, $N(x_0)$, with the property that for all x_1, x_2 in $N(x_0)$ with $x_1 < x_0 < x_2$, $g(x_1)\, g(x_2) > 0$.

It is easy to see for varisolvent families that if $F(A_1, x)$, $F(A_2, x)$ are any two distinct elements of $\mathscr{V}$, then the zeros of $F(A_1, x) - F(A_2, x)$ are either simple or double zeros.

LEMMA 1. *Let $F(A, x) \in \mathscr{V}$. Then for any $F(A_1, x) \in \mathscr{V}$ with $F(A_1, x) \not\equiv F(A, x)$, the sum of the multiplicities of the zeros of $F(A_1, x) - F(A, x)$ is at most $m(A) - 1$.*

Proof. See [15, p. 4].

As mentioned earlier the proof of the alternation theorem omitted the constant error curve possibility. The proof also needs a small modification to eliminate the possibility of a best approximation with an error curve which does not alternate and yet which is not a constant error curve. This is handled by Lemma 2.

LEMMA 2. *Let $\mathscr{V}$, $f \in C[a, b]$ and $F(A^*, x) \in \mathscr{V}$ be given. Assume that $F(A^*, x)$ is a best approximation to f on $[a, b]$ which does not give rise to a constant error curve with respect to f. Then $E(A^*, x)$ alternates one or more times.*

Proof. See [11, p. 14].

DEFINITION 10. Given f, g in $C[a, b]$, g will be called parallel to f on $[a, b]$ if $g(x) \equiv f(x) + c$ on $[a, b]$; c is some real constant.

DEFINITION 11. Let $f \in C[a, b]$. For $\epsilon > 0$,

$$N_\epsilon(f) = \{g \in C[a, b] \mid \| g(x) - f(x)\| < \epsilon\}$$

will be called an ϵ neighborhood of f.

DEFINITION 12. Let $\mathscr{V}$ be a varisolvent family on $[a, b]$, and let $f \in C[a, b]$. $\mathscr{V}$ will be said to have Property E at f if there exists an ϵ neighborhood of f, $N_\epsilon(f)$, such that for all g in $N_\epsilon(f)$ with either $g(x) \leqslant f(x)$ for all x in $[a, b]$ or $g(x) \geqslant f(x)$ for all x in $[a, b]$, a best approximation to g exists from $\mathscr{V}$ on $[a, b]$. In particular, note that Definition 12 requires that a best approximation to f exists from $\mathscr{V}$ on $[a, b]$.

We conclude with the following concept. Let B be any subset of $C[a, b]$. To say that B is an open set with respect to the Chebyshev (uniform) norm we mean that (1) either B is empty or (2) if B is nonempty, for each f in B there exists an ϵ neighborhood of f, $N_\epsilon(f)$, with $N_\epsilon(f) \subset B$.

3. MAIN THEOREM

We now present the main theorem and its proof.

THEOREM 3 (Main Theorem). *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$. Assume that $B_A = \{f \in C[a, b] \mid$ a best approximation to f exists from $\mathscr{V}$ on $[a, b]\}$ is an open set with respect to the uniform norm. Then $\mathscr{V}$ does not permit a constant error on $[a, b]$.*

The main theorem is a corollary of the following Theorem 4.

THEOREM 4. *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$ and let $f \in C[a, b]$. Assume $\mathscr{V}$ has Property E at f. Then if $F(A^*, x)$ is a best approximation to f from $\mathscr{V}$ on $[a, b]$, the error curve $E(A^*, x) = f(x) - F(A^*, x)$ must alternate at least $m(A^*)$ times.*

Before giving the proof of Theorem 4, we verify next that the main theorem is indeed a corollary. The hypothesis "B_A is an open set" guarantees that if there exists an $f \in B_A$, $\mathscr{V}$ has property E at f. Theorem 4 then ensures that if $F(A^*, x)$ is any best approximation to such an f the error curve, with respect to f, may not be a constant error curve. Since this is true for each $f \in B_A$, $\mathscr{V}$ does not permit a constant error on $[a, b]$.

Proof of Theorem 4. (Observe that at least one best approximation to f exists since $\mathscr{V}$ has Property E at f.) The proof is by contradiction. Thus, assume $E(A^*, x)$ does not alternate at least $m(A^*)$ times. Using the alternation theorem and Lemma 2, our assumption implies that $E(A^*, x) = f(x) - F(A^*, x) = C$ for all x in $[a, b]$ where C is some nonzero constant. Assume $C > 0$. (A similar argument holds if $C < 0$.) Recall that Barrar and Loeb have proved that this is impossible if $m(A^*) = 1$, 2 or 3. Thus, assume $m(A^*) \geqslant 4$.

Step 1. We construct a sequence of real valued continuous functions $\{g_n\}$ on $[a, b]$ for $n \geqslant 3$. Let K be the smallest integer with $K \geqslant ((M + 1)/2) + 1$, where M is the uniform bound for the degrees of the functions in $\mathscr{V}$. Divide $[a, b]$ into $2K$ closed intervals, $\{I_i\}_{i=1}^{2K}$, consecutively ordered, each with length $(b - a)/2K$. Set $I_i = [w_i, w_{i+1}]$ with $w_1 = a$ and $w_{2K+1} = b$. Let α be such that $0 < \alpha < 1/4((b - a)/2K)$. For $n \geqslant 3$, $g_n(x)$ is defined on I_i, for $1 \leqslant i \leqslant 2K$, as on I_i, i odd;

$$g_n = \begin{cases} \text{a continuous function from } (w_i, f(w_i) - C/2n) \text{ to} \\ (w_i + \alpha, f(w_i + \alpha)) \text{ where } g_n(x) \leqslant f(x) \text{ and} \\ |g_n(x) - f(x)| \leqslant C/2n \text{ for } x \in [w_i, w_i + \alpha] \\ f \text{ for } x \in [w_i + \alpha, w_{i+1} - \alpha] \\ \text{a continuous function from } (w_{i+1} - \alpha, f(w_{i+1} - \alpha)) \text{ to} \\ (w_{i+1}, f(w_{i+1}) - C/2n), \text{ where } g_n(x) \leqslant f(x) \text{ and} \\ |g_n(x) - f(x)| \leqslant C/2n \text{ for } x \in [w_{i+1} - \alpha, w_{i+1}] \end{cases}$$

and

on I_i, i even;

$$g_n = \begin{cases} \text{a continuous function from } (w_i, f(w_i) - C/2n) \text{ to} \\ (w_i + \alpha, f(w_i + \alpha) - C/n) \text{ where } g_n(x) < f(x) \text{ and} \\ C/2n \leqslant |g_n(x) - f(x)| \leqslant C/n \text{ for } x \in [w_i, w_i + \alpha] \\ f - C/n \text{ for } x \in [w_i + \alpha, w_{i+1} - \alpha] \\ \text{a continuous function from } (w_{i+1} - \alpha, f(w_{i+1} - \alpha) - C/n) \text{ to} \\ (w_{i+1}, f(w_{i+1}) - C/2n), \text{ where } g_n(x) < f(x) \text{ and} \\ C/2n \leqslant |g_n(x) - f(x)| \leqslant C/n \text{ for } x \in [w_{i+1} - \alpha, w_{i+1}]. \end{cases}$$

Observe that $g_n(x)$ appears as a "notch" function with K humps and K dips.

Since $\mathscr{V}$ has Property E at f, we know there exists an ϵ neighborhood of f, call it $N_{\epsilon_0}(f)$, such that for all g in $N_{\epsilon_0}(f)$, a best approximation to g exists from $\mathscr{V}$. Observe that the sequence of functions $\{g_n\}$ converges uniformly to f on $[a, b]$. Thus, there exists an N_0 such that for all $n > N_0$, g_n is in $N_{\epsilon_0}(f)$. For each $n > N_0$, we therefore know that g_n has a best approximation from $\mathscr{V}$ on $[a, b]$, call it $F(A_n, x)$. Henceforth, we shall assume n is always greater than N_0.

Clearly g_n is not in $\mathscr{V}$ since $\|f(x) - g_n(x)\| \leqslant C/3$. Consequently, $\|g_n(x) - F(A_n, x)\| \doteq C_n > 0$. Since the error curve $g_n(x) - F(A^*, x)$ is not constant and does not alternate, Lemma 2 ensures that $F(A^*, x)$ is not a best approximation to g_n. Hence, $C_n < C$. Further $C_n \geqslant C - C/n$, which is argued next. Since $F(A^*, x)$ is best to f, $C \leqslant \|f(x) - F(A_n, x)\| \leqslant \|f(x) - g_n(x)\| + \|g_n(x) - F(A_n, x)\| = C/n + C_n$.

Step 2. *Claim*: $F(A_n, x)$ is not parallel to $g_n(x)$.

Proof. Suppose it is parallel and assume first that $\|g_n(x) - F(A_n, x)\| = -(g_n(x) - F(A_n, x)) = F(A_n, x) - g_n(x) \equiv C_n$ on $[a, b]$. For all x in $[a, b]$, $|F(A_n, x) - f(x)| = |(g_n(x) - f(x)) + C_n| = (g_n(x) - f(x)) + C_n < C$ since $(g_n(x) - f(x)) \leqslant 0$. Thus $\|F(A_n, x) - f(x)\| < C$, contradiction.

Assume $g_n(x) - F(A_n, x) \equiv C_n$ on $[a, b]$. If $C_n = C - C/n$ this would imply $F(A_n, x)$ and $F(A^*, x)$ are identical on an interval and yet not equivalent on $[a, b]$, which violates Property Z. We conclude, referring to Step 1, that $C_n > C - C/n$. Thus for all x in $[a, b]$, $(g_n(x) - C) < F(A_n, x) < (g_n(x) - C) + C/n$. For i even, this inequality implies $F(A_n, x)$ crosses $F(A^*, x)$ two times on $[w_i - \alpha, w_{i+1} + \alpha]$ with the exception of possibly only one crossing on $[w_{2K} - \alpha, b]$. $F(A_n, x)$ then crosses $F(A^*, x)$ at least $2K - 1$ times. But $K \geqslant ((M + 1)/2) + 1$ implies $2K - 1 > M \geqslant m(A^*)$, violating Property Z for $F(A^*, x)$. Hence, $F(A_n, x)$ is not parallel to g_n. Lemma 2 combined with the alternation theorem guarantees that $g_n(x) - F(A_n, x)$ alternates at least $m(A_n)$ times.

Step 3. *Completion of the proof.* Our objective will be to show that for n sufficiently large, the alternation of $g_n(x) - F(A_n, x)$ gives the desired contradiction. We now make the following construction.

For notation, let $H_j = [a_j, b_j]$ where $a_j = w_{2j-1} + \alpha$ and $b_j = w_{2j} - \alpha$ for j with $1 \leqslant j \leqslant K$. Let $m(A^*) = r$. Recall that $\| g_n(x) - F(A_n, x)\| \doteq C_n$ where $C - C/n \leqslant C_n < C$. Define $L_n(x) = g_n(x) - C_n$ for all x in $[a, b]$. $L_n(x)$ is the lower bound for the error curve $E(A_n, x) = g_n(x) - F(A_n, x)$ on $[a, b]$. Note that for all n and for all x in $[a, b]$,

$$g_n(x) - C \leqslant L_n(x) \leqslant (g_n(x) - C) + C/n. \tag{1}$$

It follows easily from (1) and from the definition of $g_n(x)$, that

$$\| L_n(x) - F(A^*, x)\| \leqslant C/n \qquad \text{for all } n. \tag{2}$$

Part 1. $m(A^*) = r$ is odd. On each H_j, $1 \leqslant j \leqslant K$, select r distinct points, $\{y_i^j\}_{i=1}^r$, such that $a_j < y_1^j < y_2^j < \cdots < y_r^j < b_j$. Let $F(A^*, y_i^j) = v_i^j$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant K$ and set $\epsilon = C/4$. By varisolvency, for each j, there exists a $\delta(A^*, \epsilon, y_1^j, \ldots, y_r^j) = \delta > 0$ such that for any set of real numbers $\{z_i^j\}_{i=1}^r$, $| z_i^j - v_i^j | < \delta$ for $1 \leqslant i \leqslant r$, there exists an $F(A^j, x)$ in $\mathscr{V}$ with $F(A^j, y_i^j) = z_i^j$ for $1 \leqslant i \leqslant r$ and $\| F(A^j, x) - F(A^*, x)\| < \epsilon$. Choose $z_i^j = F(A^*, y_i^j) + 3/4\delta$ for i odd and $z_i^j = F(A^*, y_i^j) - 3/4\delta$ for i even, where $1 \leqslant i \leqslant r$. Below is a diagram of a typical $F(A^j, x)$ for any j with $1 \leqslant j \leqslant K$, where $m(A^*) = 5$. Since $F(A^j, x) - F(A^*, x)$ has $r - 1$ zeros on H_j, we know by Lemma 1 that

$$\beta_j = \min_{x \in [a, a_j] \cup [b_j, b]} [F(A^j, x) - F(A^*, x)] > 0.$$

Set $\beta_0 = \min_{1 \leqslant j \leqslant K} \beta_j > 0$.

Part 2. $m(A^*) = r$ is even. The construction of $F(A^j, x)$ on H_j for each j with $1 \leqslant j \leqslant K$ is slightly different. In this case, one chooses $y_1^j = a$,

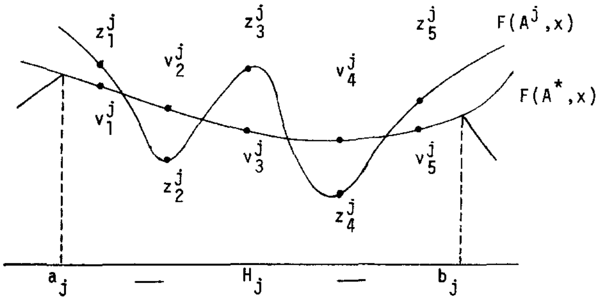

FIGURE 1.

$y_r^j = b$ and $\{y_i^j\}_{i=2}^{r-1}$ with $a_j < y_2^j < \cdots < y_{r-1}^j < b_j$. Then one takes the set of real numbers $\{z_i^j\}_{i=1}^{r}$ as $z_i^j = v_i^j = F(A^*, y_i^j)$ for $2 \leqslant i \leqslant r-1$ and $z_1^j = F(A^*, a) + \eta$, $z_r^j = F(A^*, b) + \eta$. Here η is > 0 and sufficiently small so that there exists by varisolvency, $F(A^j, x)$ in $\mathscr{V}$ with $F(A^j, y_i^j) = z_i^j$ for $1 \leqslant i \leqslant r$ and $\| F(A^j, x) - F(A^*, x)\| < \epsilon$.

Since $F(A^j, a) > F(A^*, a)$ and $F(A^j, b) > F(A^*, b)$ and since $F(A^j, x) - F(A^*, x)$ has $r-2$ zeros on H_j where $r-2$ is even, Lemma 1 shows that these zeros are simple and no other zeros may occur.

Thus, again

$$\beta_j = \min_{x \in [a, a_j] \cup [b_j, b]} [F(A^j, x) - F(A^*, x)] > 0.$$

Set $\beta_0 = \min_{1 \leqslant j \leqslant K} \beta_j > 0$.

We make the following observation. Suppose that z is a fixed even integer with $z \geqslant 6$. Then for $1 \leqslant j \leqslant K$, the oscillatory behavior of $F(A^j, x) - F(A^*, x)$ on $[a, b]$ where $m(A^*) = z$ is the same behavior as $F(A^j, x) - F(A^*, x)$ where $m(A^*) = z - 1$.

To complete the proof, recall that $E(A_n, x) = g_n(x) - F(A_n, x)$ alternates at least $m(A_n)$ times for all n. ("For all n" means of course "for all $n > N_0$.") Since the degrees of the sequence $\{F(A_n, x)\}$ are uniformly bounded by M, one can assume, by taking a subsequence if necessary, that $m(A_n) = p$ for all n. Here p is an integer with $1 \leqslant p \leqslant M$. One can see by using the continuity of $E(A_n, x)$, the definition of alternation and the compactness of $[a, b]$, that every $E(A_n, x)$ may alternate only a finite number of times. For each n, choose from the finite set of alternation points, a set of $p+1$ distinct points, $\{x_i^n\}_{i=1}^{p+1}$, $a \leqslant x_1^n < x_2^n < \cdots < x_{p+1}^n \leqslant b$, such that $E(A_n, x)$ alternates p times on this set. Let $\mathscr{L}_n$ be the set of lower alternation points from $\{x_i^n\}_{i=1}^{p+1}$. Note that for p odd, the number of points in $\mathscr{L}_n$ is $(p+1)/2$ and for p even, the number of points in $\mathscr{L}_n$ is $p/2$ or $(p/2)+1$. In either case, since $m(A_n) = p \leqslant M$ for all n and since $((M+1)/2) + 1 \leqslant K$, we have the number of points in $\mathscr{L}_n < K$ for all n. In particular this means that for any given n, there exists a j with $1 \leqslant j \leqslant K$ such that no element of $\mathscr{L}_n$ lies in the interval H_j.

Choose N_1 sufficiently large so that $n > N_1$ implies

$$C/n < \beta_0 . \tag{3}$$

For $n > N_1$, $F(A_n, x)$ has an important characteristic which we now explain. Inequalities (2) and (3) ensure that $\| L_n(x) - F(A^*, x)\| < \beta_0$, where $L_n(x) = g_n(x) - C_n$ is the lower bound for the error curve $E(A_n, x) = g_n(x) - F(A_n, x)$. Let x_0 in $[a, b]$ be a lower alternation point for $E(A_n, x)$ and observe that this means $F(A_n, x_0) = L_n(x_0)$.

But $\| L_n(x) - F(A^*, x)\| < \beta_0$ implies that if $x_0 \notin H_j$, for j with $1 \leqslant j \leqslant K$, then $F(A_n, x_0) < F(A^j, x_0)$. This is the important characteristic.

Also we claim that if x_0 is any upper alternation point of $E(A_n, x)$, then $F(A_n, x_0) > F(A^j, x_0)$ for any j with $1 \leqslant j \leqslant K$ and for any n. To see this, recall that for any j, $1 \leqslant j \leqslant K$, $\| F(A^j, x) - F(A^*, x)\| < \epsilon \doteq C/4$. In particular at x_0, $F(A^j, x_0) < F(A^*, x_0) + C/4$. However since x_0 is an upper alternation point,

$$\begin{aligned} F(A_n, x_0) &= g_n(x_0) + C_n \geqslant f(x_0) - C/n + C_n > f(x_0) \\ &> F(A^*, x_0) + C/4 > F(A^j, x_0). \end{aligned}$$

Henceforth, assume that n is a fixed $n > N_1$ and that $H_t = [a_t, b_t]$ is the associated H_j with no element of $\mathscr{L}_n$ in H_t. Recall that $F(A_n, x)$ has degree p where $1 \leqslant p \leqslant M$. We show now that for any p with $m(A_n) = p$ and $1 \leqslant p \leqslant M$, a contradiction is obtained.

Let $\{x_i^n\}_{i=1}^{p+1}$ denote the chosen set of alternation points for $E(A_n, x)$. Since $n > N_1$, each lower alternation point of $\{x_i^n\}_{i=1}^{p+1}$ has the characteristic that $F(A_n, x_i^n) < F(A^t, x_i^n)$ where x_i^n is in $\mathscr{L}_n$. Also as observed above, if x_i^n is an upper alternation point, $F(A_n, x_i^n) > F(A^t, x_i^n)$. Thus between any lower alternation point and any upper alternation point of $E(A_n, x)$, $F(A_n, x) - F(A^t, x)$ has at least one simple zero. Since $E(A_n, x)$ alternates p times on the set of $p + 1$ distinct points, $\{x_i^n\}_{i=1}^{p+1}$, this ensures that $F(A_n, x) - F(A^t, x)$ has at least p zeros on the interval $[x_1^n, x_{p+1}^n]$. This contradicts Property Z for $F(A_n, x)$ because $F(A_n, x)$ has degree p. Since this is true for any p with $m(A_n) = p$ and $1 \leqslant p \leqslant M$, there is no possible degree for $F(A_n, x)$. This contradicts the fact that $F(A_n, x)$ is a varisolvent function.

Hence, the original assumption that $E(A^*, x)$ does not alternate at least $m(A^*)$ times, has led to a contradiction. This completes the proof of Theorem 4.

We add here a few comments on the assumption in the main theorem that B_A is an open set with respect to the uniform norm. First we introduce two lemmas and define a closure property, called Property C.

LEMMA 3. *Given $\mathscr{V}$ on $[a, b]$, let $\{F(A_n, x)\}$ be a sequence of uniformly bounded functions on $[a, b]$ from $\mathscr{V}$. Then this sequence contains a pointwise convergent subsequence on $[a, b]$.*

Proof. See [15, p. 6].

DEFINITION 13. Let $\mathscr{V}$ on $[a, b]$ be given. $\mathscr{V}$ will be said to have the closure property, Property C, if for every uniformly bounded sequence $\{F(A_n, x)\}$ from $\mathscr{V}$, there exists at least one pointwise convergent subsequence of $\{F(A_n, x)\}$ which converges pointwise on a dense subset of $[a, b]$ to an element in $\mathscr{V}$.

We note that Lemma 3 and Property C comprise Dunham's definition of dense compactness for continuous functions on an interval (see [8, p. 444]).

LEMMA 4. *Let $\mathscr{V}$ have Property C and let $f \in C[a, b]$. Then a best approximation to f from $\mathscr{V}$ on $[a, b]$ exists.*

Proof. See [8, p. 444].

The reason for introducing Property C is that it appears to be a useful criterion to determine whether the hypothesis of the main theorem is satisfied for a given varisolvent family. In particular, Lemma 4 guarantees that if $\mathscr{V}$ has Property C, the set B_A is an open set for $\mathscr{V}$. We illustrate with some examples.

Tornheim's Theorem 5 in [17] guarantees that unisolvent families possess Property C. Rice's results in [14, pp. 74–80] demonstrate that the family of rational functions $R_{n,m}$ has Property C ($R_{n,m}$ is defined in Part 5). One can show directly that other varisolvent families such as $\mathscr{V} = \{F(A, x) = a/(1 + ax) \mid -1 < a < 1, x \in [-1, 1]\}$ with $m(A) = 1$ have Property C.

However, there are varisolvent families which do not possess Property C. A simple example of such a family is $\mathscr{V} = \{F(A, x) \mid F(A, x) \equiv C, C < 0\}$, where $m(A) = 1$. (For example, consider the sequence $\{-1/n\}$.) Yet, as can be seen from the following lemma, $\mathscr{V}$ satisfies the hypothesis of the main theorem.

LEMMA 5. *Let $\mathscr{V}$ be varisolvent on $[a, b]$. Then the set $\{f \in C[a, b] \mid f$ has a best approximation from $\mathscr{V}$ on $[a, b]$ of maximal degree$\}$ is an open set with respect to the uniform norm.*

Proof. This follows from the theorem in [7, p. 607] and a remark contained in its proof.

The family $\mathscr{V}$ mentioned above is a typical example of a locally unisolvent family, i.e., a varisolvent family of fixed degree. In particular, each $F(A, x) \in \mathscr{V}$ has maximal degree. Thus, Lemma 5 guarantees that $\mathscr{V}$, and in fact any locally unisolvent family, satisfies the hypothesis "B_A is an open set."

Lemma 5 also guarantees the following. Assume $f \in C[a, b]$, $\mathscr{V}$ is a varisolvent family and $F(A^*, x)$ is a best approximation to f from $\mathscr{V}$ on $[a, b]$ of maximal degree. Then Lemma 5 guarantees that $\mathscr{V}$ has Property E at f and thus by Theorem 4 cannot give rise to a constant error curve. Therefore, Braess's result [3], that a function of maximal degree may not give rise to a constant error curve, follows from Theorem 4.

The only known varisolvent family to which the main theorem may not apply seems to be "sums of exponentials." By this we mean the family $E_n^0 = \{F(A, x) = \sum_{i=1}^{n} a_i \exp(\lambda_i x) \mid a_i, \lambda_i \text{ real}, \prod_{i=1}^{r} a_i \neq 0, \lambda_1 > \lambda_2 > \cdots > \lambda_r$, all λ_i mutually distinct for $1 \leqslant i \leqslant n\}$. It appears to be an open question as to whether B_A is an open set for E_n^0.

4. Applications

a. *Unisolvent and Locally Unisolvent Families*

Tornheim's Theorem 5 in [17] shows that pointwise convergence for unisolvent families is in fact uniform. Using this uniform convergence, the proof of Theorem 4 becomes relatively short. The proof is in fact complete at the end of step 2, with the following comments added. The sequence $\{F(A_n, x)\}$ is such that $E(A_n, x)$ alternates for all n. Since a unisolvent family has Property C, $\{F(A_n, x)\}$ has a subsequence, call it again $\{F(A_n, x)\}$, converging uniformly to some $F(A, x)$ in $\mathscr{U}$. One can easily show $F(A, x)$ is a best approximation to f. But since the convergence of $\{F(A_n, x)\}$ is uniform and $E(A_n, x)$ alternates for all n, $F(A, x)$ is not identical to $F(A^*, x)$ or $f(x) + C$. Therefore $f(x) - F(A, x)$ alternates, contradicting the following lemma.

Lemma 6. *Given $\mathscr{V}$ on $[a, b]$, $f \in C[a, b]$ and $F(A_1, x)$ and $F(A_2, x)$ best approximations to f from $\mathscr{V}$ on $[a, b]$. It is impossible to have one of $F(A_1, x)$ and $F(A_2, x)$ give rise to a constant error curve with respect to f and the other give rise to an alternating error curve with respect to f.*

Proof. Easy zero counting argument using Lemma 1 (see [11, p. 32]).

One can also show that the same short proof holds for locally unisolvent families which have Property C.

We conclude application (a) with a comment on uniqueness. For a varisolvent family with B_A an open set we can now guarantee that if a best approximation exists, it is unique. This is true because the main theorem combined with Lemma 2 and the alternation theorem ensures that the error curve associated with a best approximation must alternate. Uniqueness then follows by a zero counting argument as appears in [15, p. 12]. If $\mathscr{V}$ is such that B_A is not an open set, Lemma 6 asserts that the only possibility of non-uniqueness is

$$F(A_1, x) \equiv f(x) + C \qquad \text{and} \qquad F(A_2, x) \equiv f(x) - C, \qquad C > 0.$$

b. *A Betweenness Property*

THEOREM 5. *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$ that does not permit a constant error. Assume $F(A_1, x)$, $F(A_2, x)$ in $\mathscr{V}$ are such that $F(A_1, x) < F(A_2, x)$ on $[a, b]$. Then there exists an $F(A_0, x)$ in $\mathscr{V}$ with $F(A_1, x) < F(A_0, x) < F(A_2, x)$ on $[a, b]$.*

Note that if the set B_A for $\mathscr{V}$ is an open set with respect to the uniform norm, Theorem 5 is an application of the main theorem. We have used the assumption, "$\mathscr{V}$ does not permit a constant error," so that Theorem 5 is as general as possible.

Proof. Let $\min_{x \in [a,b]} [F(A_2, x) - F(A_1, x)] \doteq m > 0$. Let n be sufficiently large so that $1/n < m/2$. Define $f(x) = F(A_1, x) + 1/n$ and notice that f is continuous with $F(A_1, x) < f(x) < F(A_2, x)$ on $[a, b]$. If there exists an A_0 in P such that $F(A_0, x) \equiv f(x)$ on $[a, b]$, then the theorem is proved. So assume $f \notin \mathscr{V}$.

Since $\mathscr{V}$ does not permit a constant error, we know that

$$\inf_{A \in P} \| f(x) - F(A, x)\| \doteq e < 1/n.$$

Set $1/n - e = e_1 > 0$. By definition of infimum, there exists an $F(A_0, x)$ in $\mathscr{V}$ with $\| f(x) - F(A_0, x)\| < e + e_1/2$. Since $e + e_1/2 < 1/n$, we have $F(A_1, x) < F(A_0, x) < F(A_2, x)$ on $[a, b]$. We use a similar argument to show that a constant error may not arise in simultaneous approximation.

c. *Simultaneous Approximation*

The simultaneous problem as described by Dunham in [4] is the following. Let f^+ and f^- be given real valued functions on $[a, b]$ with f^+ upper semicontinuous, f^- lower semicontinuous and $f^-(x) \leqslant f^+(x)$ for all x in $[a, b]$. For any real valued function, $g(x)$, define $\| g(x)\| = \sup_{x \in [a,b]} | g(x)|$. Note that since $f^-(x) \leqslant f^+(x)$, the sets

$$\{x \in [a, b] \mid f^+(x) = -\infty\} \qquad \text{and} \qquad \{x \in [a, b] \mid f^-(x) = +\infty\}$$

are empty. Given an approximating family $\mathscr{F}$, the simultaneous problem is to find $F(A, x)$ in $\mathscr{F}$ to minimize $e(A) = \max\{\|f^+(x) - F(A, x)\|, \|f^-(x) - F(A, x)\|\}$. If $F(A^*, x)$ in $\mathscr{F}$ is such that $e(A^*) = \inf_{A \in P} e(A)$, we say that $F(A^*, x)$ is a best simultaneous approximation.

In [4], $x_0 \in [a, b]$ is called a straddle point if there exists a parameter A with

$$f^+(x_0) - F(A, x_0) = e(A) = F(A, x_0) - f^-(x_0). \tag{1}$$

It is easy to see that $F(A, x)$ is then a best simultaneous approximation. We shall use the terminology, x_0 is a straddle point for $F(A, x)$, if (1) holds for $F(A, x)$. For each $F(A, x)$ in $\mathscr{F}$ set $E^+(A, x) = f^+(x) - F(A, x)$ and $E^-(A, x) = f^-(x) - F(A, x)$. The symbol $p(j)$ will stand for the sign $+$ if j is even and for the sign $-$ if j is odd.

DEFINITION 14. An approximation $F(A, x) \in \mathscr{F}$ to f^+ and f^- on $[a, b]$ will be said to have k alternations ($k \geqslant 1$) on $[a, b]$ if there exist $k + 1$ distinct points in $[a, b]$, $\{x_i\}_{i=1}^{k+1}$ with $a \leqslant x_1 < x_2 < \cdots < x_{k+1} \leqslant b$ and an integer $j = 0$ or 1 such that $E^{p(i+j)}(A, x_i) = (-1)^{i+j} e(A)$, $1 \leqslant i \leqslant k + 1$.

DEFINITION 15. Let $F(A^*, x) \in \mathscr{F}$ be a best simultaneous approximation to f^+ and f^- on $[a, b]$. $F(A^*, x)$ will be said to give rise to a constant error curve with respect to f^+ and f^- if $\max\{f^+(x) - F(A^*, x), F(A^*, x) - f^-(x)\} \equiv e(A^*)$.

For each $F(A, x)$ in $\mathscr{F}$ define $f_R(x) = f^+(x) - e(A)$ and $f_G(x) = f^-(x) + e(A)$. It is easy to show that for all x in $[a, b]$, $f_R(x) \leqslant F(A, x) \leqslant f_G(x)$. It is useful to observe and also easily shown that x_0 is a straddle point for $F(A, x)$ iff $f_R(x_0) = f_G(x_0)$. Thus if one assumes $F(A, x)$ has no straddle points, then $f_R(x) < f_G(x)$ holds for all x in $[a, b]$. On inspecting Definition 15, one concludes that if a straddle point for $F(A, x)$ does not occur, then $F(A, x)$ gives rise to a constant error curve iff $F(A, x) = f_R(x)$ for all x in $[a, b]$ or $F(A, x) = f_G(x)$ for all x in $[a, b]$.

For a given $F(A, x)$ in $\mathscr{F}$, define x in $[a, b]$ to be a $+$ point if $F(A, x) = f_R(x)$ and a $-$ point if $F(A, x) = f_G(x)$. M^+ denotes the set of $+$ points and M^- the set of $-$ points for $F(A, x)$.

Assume for the moment that $F(A^*, x)$ is a best simultaneous approximation. It is noted in [4] that the set $M^+ \cup M^-$ is not empty for $F(A^*, x)$. However the possibility that one of the sets M^+ and M^- might be empty does not appear to be ruled out. Suppose, for example, M^- is empty. Then one possibility is that $F(A^*, x) = f_R(x)$ for all x in $[a, b]$, i.e., $F(A^*, x)$ gives rise to a constant error curve.

With the exception that one of the sets M^+ and M^- might be empty, the following alternation theorem proven in [4] holds.

THEOREM 6. *Let $\mathscr{U}$ be a unisolvent family of degree n on $[a, b]$. Then $F(A^*, x)$ is a best approximation to f^+ and f^- on $[a, b]$ from $\mathscr{U}$ iff $F(A^*, x)$ has a straddle point or n alternations on $[a, b]$.*

Proof. See [4, p. 474].

Further it is mentioned that Theorem 6 is easily generalized to varisolvent families.

The next lemma rules out for varisolvent families which do not permit a constant error, the possibility that one of the sets M^+ and M^- might be empty.

LEMMA 7. *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$ which does not permit a constant error. Assume $F(A^*, x) \in \mathscr{V}$ is a best simultaneous approximation to f^+ and f^- on $[a, b]$. Then the associated sets M^+ and M^- are both nonempty.*

Note that Lemma 7 is an application of the main theorem if the set B_A is an open set for $\mathscr{V}$.

Proof. Observe that if $F(A^*, x)$ has a straddle point, x_0, then x_0 is both a $+$ and a $-$ point. Thus, x_0 is in both M^+ and M^-. So assume $F(A^*, x)$ has no straddle points. This implies that $f_R(x) < f_G(x)$ on $[a, b]$. Suppose M^- is empty. (A similar argument holds if M^+ is empty.) M^- empty means that $F(A^*, x) < f_G(x)$ on $[a, b]$. Since $f_G(x)$ is lower semicontinuous, we have $\min_{x \in [a,b]} [f_G(x) - F(A^*, x)] \doteq m > 0$.

Case 1. $F(A^*, x) \not\equiv f_R(x)$ on $[a, b]$.

In this case, there exists x_1 in $[a, b]$ with $F(A^*, x_1) > f_R(x_1)$. Since $f_R(x)$ is upper semicontinuous, there is a neighborhood of x_1, $N(x_1)$, with $F(A^*, x) > f_R(x)$ for all x in $N(x_1)$. An argument similar to the one used in Lemma 2 can be applied to construct a better simultaneous approximation than $F(A^*, x)$. Contradiction.

Case 2. $F(A^*, x) \equiv f_R(x)$ on $[a, b]$.

(This is the case of a constant error curve for simultaneous approximation when straddle points do not occur.) Choose n sufficiently large so that $1/n < m/2$. Consider the continuous function $F(A^*, x) + 1/n$. $F(A^*, x) + 1/n$ is not in $\mathscr{V}$ since otherwise it would be a better simultaneous approximation than $F(A^*, x)$. Further $F(A^*, x)$ is parallel to $F(A^*, x) + 1/n$. Since $\mathscr{V}$ does not permit a constant error, we conclude that $F(A^*, x)$ is not a best approximation to $F(A^*, x) + 1/n$ from $\mathscr{V}$. Therefore,

$$\inf_{A \in P} \|(F(A^*, x) + 1/n) - F(A, x)\| \doteq e < 1/n.$$

Let $e_1 = 1/n - e > 0$. By definition of infimum, there exists an $F(A, x)$ in $\mathscr{V}$ with $\|(F(A^*, x) + 1/n) - F(A, x)\| < e + e_1/2$. Since $e + e_1/2 < 1/n$, it follows that $f_R(x) < F(A, x)$ on $[a, b]$ and that $F(A, x) < f_G(x)$ on $[a, b]$. But then $e(A) < e(A^*)$ holds, implying that $F(A, x)$ is a better simultaneous approximation than $F(A^*, x)$. Contradiction.

d. *Restricted Range Approximation*

Let an approximating family $\mathscr{F}$ be given. In [9] Dunham considered approximating $f \in C[a, b]$ from members of $\mathscr{F}$ where each member $F(A, x)$ satisfies $\ell(x) \leqslant F(A, x) \leqslant u(x)$ on $[a, b]$. ℓ is an upper semicontinuous mapping into the extended real line while u is a lower semicontinuous mapping into the extended real line, such that $\ell(x) < u(x)$ on $[a, b]$. Dunham also assumes that $\ell(x) \leqslant f(x) \leqslant u(x)$ on $[a, b]$. Replacing the approximating family studied by Dunham (one requiring Property A and Property Z) by a varisolvent family satisfying the hypothesis of our main theorem produces the same alternation theorem and uniqueness corollary obtained by Dunham.

The main theorem can also be applied when the condition $\ell(x) \leqslant f(x) \leqslant u(x)$ is deleted and to certain generalized weight function approximation problems (see Tornga [16]).

e. *Approximation on a Proper, Nonempty, Compact Subset of* $[a, b]$

LEMMA 8. *Let $\mathscr{V}$ be a varisolvent family on $[a, b]$ and let X be any proper, nonempty compact subset of $[a, b]$. Then $\mathscr{V}$ does not permit a constant error on X.*

Proof. If X is an interval, Lemma 8 follows directly from Theorem 2 of Meinardus and Taylor. If it does not, the result follows by a simple argument, similar to the one used in the proof of Theorem 2.

Using Lemma 8 and the fact that Lemma 2 can be generalized to X, one obtains that the alternation theorem holds for any proper, nonempty compact subset of $[a, b]$.

5. EXAMPLES

As mentioned in the introduction, the other main approach which eliminates the constant error possibility is that of Meinardus and Schwedt and of Barrar and Loeb. We present here a few examples of varisolvent families to which the main theorem applies but their approach does not.

The examples are based on Kaufman and Belford's paper, [10]. They prove the following theorem.

THEOREM 7. *Let X be a compact metric space and $\mathscr{V}$ a varisolvent family of functions on X. Let $W(x, y)$ satisfy* (a) *$W(x, y)$ is a strictly increasing* *function of y for every $x \in X$;* (b) *$W(x, y)$ is continuous on $X \times (-\infty, \infty)$. Then $\mathscr{W} = \{W(x, F(A, x)) \mid F(A, x) \in \mathscr{V}\}$ is a varisolvent family on X, each $W(x, F(A, x))$ having the same degree as the corresponding $F(A, x)$.*

Also a short proof shows that if we assume (c) $\lim_{|y|\to\infty} |W(x, y)| = \infty$, *then if $\mathscr{V}$ has Property C, so does $\mathscr{W}$.*

For the examples we use the varisolvent family $R_{n,m} = \{F(A, x) = \sum_{k=0}^{n} p_k x^k / \sum_{k=0}^{m} q_k x^k = P(x)/Q(x) \mid F(A, x)$ is reduced to lowest terms, $Q(x) \neq 0$ for all x in $[a, b]$, p_k, q_k real$\}$. For $F(A, x) \in R_{n,m}$, the degree of $F(A, x)$ is defined by

$$m(A) = \begin{cases} n+1 & \text{if } F(A,x) \equiv 0, \\ \max\{n + \partial Q, m + \partial P\} + 1 & \text{if } F(A,x) \not\equiv 0, \end{cases}$$

where ∂Q, ∂P are the degrees of the polynomials. Recall that $R_{n,m}$ has Property C on $[a, b]$.

We note here that $R_{n,m}$ may itself not satisfy the "derivative approach" of Meinardus and Schwedt and of Barrar and Loeb. Usually one requires $F(A, x) \in R_{n,m}$ to be reduced to lowest terms. This reduction however entails identifying or selecting points in R^{n+m+2} space. On this basis it does not appear obvious that the parameter space P can be chosen to be open as is required. The openness of P is in fact essential in Barrar and Loeb's proof that a constant error may not arise in the "derivative approach."

EXAMPLE 1. Let $W(x, y) = y^{1/3}$, $[a, b] = [0, 1]$ and $\mathscr{V} = R_{2,2}$. (We observe that $W(x, y) = y^{k/j}$, k and j odd, is a transformation (order function) suggested by Dunham [5].) By Theorem 7,

$$\mathscr{W} = \{W(x, F(A, x)) \mid F(A, x) \in R_{2,2}\}$$

is a varisolvent family. Further it has Property C since $\lim_{|y|\to\infty} |W(x, y)| = \infty$. Thus, the main theorem applies, guaranteeing that $\mathscr{W}$ does not permit a constant error. However the assumptions of Meinardus and Schwedt and of Barrar and Loeb are not satisfied by $\mathscr{W}$. For example, take $F(A, x) = (-\frac{1}{2} + x)/(1 + x) \doteq (p_0 + p_1 x + p_2 x^2)/(q_0 + q_1 x + q_2 x^2)$ which has degree $m(A) = 4$. Then

$$\begin{aligned} \partial W(x, F(A, x))/\partial p_0 &= \frac{1}{3} \cdot \frac{1}{\{(-\frac{1}{2} + x)/(1 + x)\}^{2/3}} \cdot \frac{1}{1 + x} \\ &= \frac{1}{3} \cdot \frac{1}{(1 + x)^{1/3}} \cdot \frac{1}{(-\frac{1}{2} + x)^{2/3}} \end{aligned}$$

which does not exist at $x = \frac{1}{2}$.

The Main Theorem is in fact the only applicabıe theory. The results of Barrar and Loeb, where the degree of a best approximation is 1, 2, or 3 cannot be applied to $F(A, x)$ since $m(A) = 4$. Also the result of Braess, is not applicable since $m(A)$ is not maximal in $R_{2,2}$.

EXAMPLE 2. Let

$$W(x, y) = \begin{cases} y & \text{for } \ y \in (-\infty, 2], \\ (\tfrac{1}{2}) y^2 & \text{for } \ y \in [2, \infty), \end{cases}$$

$[a, b] = [1, 3]$, $\mathscr{V} = R_{4,2}$, $\mathscr{W} = \{W(x, F(A, x)) \mid F(A, x) \in R_{4,2}\}$ and $F(A, x) = x^3/(2 + x)$ with $m(A) = 6$. Again $\mathscr{W}$ has Property C and the main theorem applies. Here

$$W(x, F(A, x)) = \begin{cases} x^3/(2 + x) & \text{for } \ x \in [1, 2], \\ (\tfrac{1}{2})[x^3/(2 + x)]^2 & \text{for } \ x \in [2, 3]. \end{cases}$$

Thus,

$$\partial W(x, F(A, x))/\partial q_0 = \begin{cases} -x^3/(2 + x)^2 & \text{for } \ x \in [1, 2), \\ [x^3/(2 + x)][-x^3/(2 + x)^2] & \text{for } \ x \in (2, 3], \end{cases}$$

which does not exist at $x = 2$. As in Example 1, only the main theorem applies since all the derivatives with respect to the parameters do not exist, since $m(A) > 3$ and since $m(A)$ is not maximal.

EXAMPLE 3. Let $F(A, x)$ be an element of $\mathscr{F}$ where $\mathscr{F}$ satisfies the assumptions (1)–(4) of Meinardus and Schwedt and of Barrar and Loeb. In particular, recall that the linear span of $H(A)$ is a Haar space of dimension $d(A)$. Kaufman and Belford show that if (a') $W(x, y)$ is continuously differentiable with respect to y and (b') $\partial W(x, y)/\partial y > 0$ for all $(x, y) \in X \times (-\infty, \infty)$, then for each $F(A, x)$ in $\mathscr{F}$, the linear span of $H_W(A) \doteq \{\partial W(x, F(A, x))/\partial a_i \mid 1 \leqslant i \leqslant s\}$ is a Haar space of dimension $d(A)$. In general, one can show that for any $W(x, y)$ satisfying (a'), the dimension of the linear span of $H_W(A)$, $d_W(A)$, is less than or equal to $d(A)$.

Let $W(x, y) = y^3$ and note that $\partial W(x, y)/\partial y = 3y^2 = 0$ at $y = 0$. Set $\mathscr{F} = R_{3,1}$, $[a, b] = [-1, 1]$, $\mathscr{W} = \{W(x, F(A, x)) \mid F(A, x) \in R_{3,1}\}$ and $F(A, x) = (\frac{1}{2} - x^2)/1$. Since $\mathscr{F}$ is a varisolvent family, $d(A) = m(A)$ which is 4 here. Now $\partial W(x, F(A, x))/\partial p_0 = 3(\frac{1}{2} - x^2)^2$, $\partial W(x, F(A, x))/\partial p_1 = 3(\frac{1}{2} - x^2)^2 x$, and so forth. One nonzero linear combination of $H_W(A)$ is $3(\frac{1}{2} - x^2)^2 x^2 - 3(\frac{1}{2} - x^2)^2 1 = 3(\frac{1}{2} - x^2)^2(x^2 - 1)$, which has four zeros; $x = 1, -1, 1/\sqrt{2}, -1/\sqrt{2}$, on $[a, b]$. Since $d_W(A) \leqslant d(A) = 4$, the existence of these four zeros implies that the linear span of $H_W(A)$ is not a Haar space. Thus again, the "derivative approach" cannot be used, $m(A) > 3$ and $m(A)$ is not maximal. But the main theorem applies since $\mathscr{W}$ has Property C.

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